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Quantized spin Hall effect in ^{3}He-A and other p-wave paired Fermi systems

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In this paper, we propose the quantized spin Hall effect (SHE) in the vortex state of a rotating p-wave paired Fermi system in an inhomogeneous magnetic field and in a weak periodic potential. It is the three-dimensional extension of the spin Hall effect for a ^{3}He-A superfluid film previously studied in the literature. It may also be considered as a generalization of the three-dimensional quantized charge Hall effect of Bloch electrons in the work previously studied in the literature to the spin transport. The A phase of ^{3}He or, more generally, the p-wave paired phase of a cold Fermi atomic gas under suitable conditions should be a good candidate to observe the SHE because the system has a conserved spin current (with no spin-orbit couplings).

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I. INTRODUCTION

Recently, spin transport in condensed matter systems has received wide attention. One of the most striking phenomena is the spin Hall effect (SHE) in semiconductors.[3] In this effect, the spin current is driven by an electric field and flows in a direction perpendicular to it. Time reversal symmetry is respected; the situation is remarkably different from the Hall effect of charged currents in a magnetic field. However, the SHE in semiconductors needs appreciable spin-orbit couplings, which make the spin current nonconserved. This causes several complications in theoretical study and makes the relationship between the spin current and the experimentally observable spin accumulation intricate and obscure.

In literature, another class of condensed matter systems, namely, superfluids, has been suggested, in which the SHE may be observable without the need of spin-orbit couplings. It has been shown that in the vortex state of two-dimensional d-wave superconductors, the spin current can be generated by an inhomogeneous magnetic field and flows perpendicular to the field gradient.[4] Moreover, a similar spin Hall effect was suggested for a rotating superfluid ^{3}He film in a nonuniform magnetic field.[1] In this case, time reversal symmetry is broken. If the quasiparticle spectrum has a gap and the Fermi level lies within the gap, the spin Hall conductance can be expressed as a topological number (the first Chern number) and therefore is quantized, which is in units of $\mu_{\mathrm{He}}/8\pi$ with $\hbar=c=1$, where μ_{He} is the Zeeman coupling constant for the ^{3}He atom. This effect is thus the spin analog of the quantized charge Hall effect of Bloch electrons.[5,6] The distinctive feature of the SHE in superfluids is the absence of spin-orbit couplings. In general, spin-orbit couplings exist in superconductors or semiconductors because of the presence of the internal crystal fields or confining fields (in the case of heterostructures). However, spin-orbit couplings do not exist in superfluid ^{3}He, of which the constituents are neutral atoms. (Recall that spin-orbit couplings exist only for charged particles.) Superfluid ^{3}He is in the spin-triplet p-wave state[7] and the spin rotational symmetry is spontaneously broken. However, one of ^{3}He superfluid phases, i.e., the ^{3}He-A phase, has a residual rotational symmetry around the z axis in the spin space. In this phase, the system has a conserved spin current.[1,7] Therefore, the ^{3}He-A film should be a good candidate to observe the quantized spin Hall effect (QSHE) without worrying about complications due to nonconservation of the spin current. This motivates us to study this system in the more general three-dimensional cases.

In this paper, we propose the quantized SHE in the vortex state of a rotating p-wave paired Fermi system in a nonuniform magnetic field and in a periodic potential in three dimensions. Our proposal is a generalization of the SHE for a thin film of the ^{3}He-A superfluid studied in Ref. 1. On the other hand, it is the spin analog of the three-dimensional quantum charge Hall effect (3DQHE) for Bloch electrons suggested in Ref. 2. Our discussions are also valid for the vortex state of a cold Fermi atomic gas that is in the p-wave pairing regime.[8]

II. VORTEX STATE IN THE ^{3}He-A PHASE

We use the same notations as in Ref. 1. The gap matrix for a ^{3}He superfluid is defined as $\hat{\Delta}_{\alpha\beta}(\mathbf{x},\mathbf{y}) = \Sigma_{\gamma\delta} V_{\alpha\beta\gamma\delta}(\mathbf{x},\mathbf{y})\langle\psi_\gamma(\mathbf{x})\psi_\delta(\mathbf{y})\rangle$, where $\psi_\alpha(\mathbf{x})$ is the fermion field for the ^{3}He atom with spin $\alpha=\uparrow,\downarrow$, $\langle\cdots\rangle$ is the thermal expectation value, and $V_{\alpha\beta\gamma\delta}(\mathbf{x},\mathbf{y})$ is the pairing interaction. Under the U(1) gauge transformation $\psi_\alpha(\mathbf{x})\to e^{-i\theta(\mathbf{x})}\psi_\alpha(\mathbf{x})$, the gap matrix is transformed as $\hat{\Delta}_{\alpha\beta}(\mathbf{x},\mathbf{y})\to\hat{\Delta}_{\alpha\beta}(\mathbf{x},\mathbf{y})e^{-i\theta(\mathbf{x})-i\theta(\mathbf{y})}$. The matrix can be written in a form $\hat{\Delta}_{\alpha\beta}(\mathbf{x},\mathbf{y})=\Phi(\mathbf{R}_G)\int d^3k/(2\pi)^3\,\hat{\Psi}_{\alpha\beta}(\mathbf{k})e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$, where $\Phi(\mathbf{R}_G)$ denotes the gap amplitude depending on the center of mass of a Cooper pair $\mathbf{R}_G=(\mathbf{x}+\mathbf{y})/2$ and $\hat{\Psi}_{\alpha\beta}(\mathbf{k})$ the order parameter. For the spin-triplet pairing state, the order parameter is written by using the $\mathbf{d}(\mathbf{k})$ vector in the spin space[7] as

$\hat{\Psi}_{\alpha\beta}(\mathbf{k})=i\mathbf{d}(\mathbf{k})\cdot(\sigma\sigma_2)_{\alpha\beta}$, where $\sigma=(\sigma_1,\sigma_2,\sigma_3)$ is the Pauli matrix.

In the ^{3}He-A phase, the $\mathbf{d}$ vector is given by

$$\mathbf{d}(\mathbf{k})=\mathbf{e}_z(k_x+ik_y). \quad (1)$$

Clearly, the order parameter is invariant under rotations around the z axis in the spin space. As we mentioned, the presence of the spin rotational symmetry is crucial for defining a conserved spin current. In the particle-hole representation, the quasiparticles are described by a two-component field, i.e.,

$$\Psi(\mathbf{x})=\begin{pmatrix}\psi_\uparrow(\mathbf{x})\\ \psi_\downarrow^*(\mathbf{x})\end{pmatrix}.$$

The Hamiltonian for a rotating superfluid with angular velocity $\mathbf{\Omega}$ in the rotating frame reads[1]

$$H=\int d^3xd^3y\Psi^\dagger(\mathbf{x})\mathcal{H}(\hat{\mathbf{p}},\mathbf{x},\mathbf{y})\Psi(\mathbf{y}), \quad (2)$$

with

$$\mathcal{H}(\hat{\mathbf{p}},\mathbf{x},\mathbf{y})=\begin{pmatrix}\epsilon(\hat{\mathbf{p}}-m\mathbf{R})\delta^3(\mathbf{x}-\mathbf{y}) & \Delta_A^{\mathrm{inv}}(\mathbf{x},\mathbf{y})e^{-(i/2)[\varphi(\mathbf{x})+\varphi(\mathbf{y})]}\\ -\Delta_A^{\mathrm{inv}*}(\mathbf{x},\mathbf{y})e^{(i/2)[\varphi(\mathbf{x})+\varphi(\mathbf{y})]} & -\epsilon(\hat{\mathbf{p}}+m\mathbf{R})\delta^3(\mathbf{x}-\mathbf{y})\end{pmatrix},$$

$$\epsilon(\hat{\mathbf{p}}\pm m\mathbf{R})=\frac{1}{2m}(\hat{\mathbf{p}}\pm m\mathbf{R})^2-\mu_F,$$

$$\hat{\mathbf{p}}=-i\nabla,$$

$$\mathbf{R}=\mathbf{\Omega}\times\mathbf{x},$$

where $\Delta_A^{\mathrm{inv}}(\mathbf{x},\mathbf{y})$ is the gauge invariant part of the spin-traced-out gap function for the A-phase $\Delta_A(\mathbf{x},\mathbf{y})\equiv(1/2)\mathrm{Tr}[\hat{\Delta}(\mathbf{x},\mathbf{y})\sigma_1]$ and $[\varphi(\mathbf{x})+\varphi(\mathbf{y})]/2$ is the gauge dependent phase. Namely, we have a relation $\Delta_A(\mathbf{x},\mathbf{y})=\Delta_A^{\mathrm{inv}}(\mathbf{x},\mathbf{y})\exp\{-i[\varphi(\mathbf{x})+\varphi(\mathbf{y})]/2\}$. Moreover, since $\Delta_A^{\mathrm{inv}}(\mathbf{x},\mathbf{y})$ depends only on the relative coordinate $\mathbf{x}-\mathbf{y}$, for p-wave pairing, we have $\Delta_A^{\mathrm{inv}*}(\mathbf{y},\mathbf{x})=-\Delta_A^{\mathrm{inv}*}(\mathbf{x},\mathbf{y})$. From the above equations, the correspondence between the case of a superconductor in an applied magnetic field and that of a rotating superfluid is clear, as shown in Table I. From the correspondence, one naturally expects that a vortex state emerges when Ω is larger than a certain critical value Ω_{c1}. The vortex lines are parallel to $\mathbf{\Omega}$, and the vortices form a lattice on the plane perpendicular to $\hat{\mathbf{n}}=\mathbf{\Omega}/\Omega$. The center $\mathbf{x}^i$ of the ith vortex can be expressed in terms of the primitive vectors $\mathbf{a}$ and $\mathbf{b}$ of the lattice as

$$\mathbf{x}^i=l_i\mathbf{a}+n_i\mathbf{b}+z\hat{\mathbf{n}} \quad (l_i,n_i \text{ integers, } z \text{ real}),$$

where the phase $\varphi(\mathbf{x})$ of the gap function satisfies

TABLE I. The correspondence between a superconductor in an applied magnetic field and a rotating superfluid.

Superconductors	The rotating superfluid
e	m
$\mathbf{A}=\frac{1}{2}\mathbf{B}\times\mathbf{r}$	$\mathbf{R}=\mathbf{\Omega}\times\mathbf{r}$
$\mathbf{B}$	$2\mathbf{\Omega}$

$$\nabla\times\nabla\varphi(\mathbf{x})=2\pi\hat{\mathbf{n}}\sum_i\delta^3(\mathbf{x}-\mathbf{x}^i). \quad (3)$$

Since a Cooper pair has a "charge" $2m$, a vortex has a "flux" π/m and one obtains

$$2\Omega=\frac{\pi}{m|\mathbf{a}\times\mathbf{b}|}.$$

This relation is consistent with the correspondence between the Ω flux in the unit cell of the vortex lattice and the flux quanta π/e in superconductors, which is a half of the unit flux $2\pi/e$. The difference is due to the fact that the charge of a Cooper pair in superconductors is $2e$.

Now, we proceed to examine the lattice symmetry of this vortex state. We will see that the system has a "magnetic" translational symmetry with respect to *two cells* of the vortex lattice since a vortex has a flux π/m, which is a half of the "unit flux" $2\pi/m$. In an analogy to the system of charge particles on a lattice, we consider the magnetic translation operator as[1]

$$T_{\delta\mathbf{r}}=\exp[i\delta\mathbf{r}\cdot(\hat{\mathbf{p}}+\tau_3m\mathbf{R})],$$

where τ_3 is the third component of the Pauli matrices in a particle-hole space. By using the gauge degrees of freedom, one can introduce the following constraints without conflict with Eq. (3) (Ref. 1):

$$\varphi(\mathbf{x}+\mathbf{a})=\varphi(\mathbf{x})-m\mathbf{a}\cdot\mathbf{R},$$

$$\varphi(\mathbf{x}+\mathbf{b}) = \varphi(\mathbf{x}) - m\mathbf{b}\cdot\mathbf{R}.$$

Then, we see that the system has the translational symmetry generated by $T_{\mathbf{a}}$ and $T_{2\mathbf{b}}$,

$$[H,T_{\mathbf{a}}] = [H,T_{2\mathbf{b}}] = 0, [T_{\mathbf{a}},T_{2\mathbf{b}}] = 0.$$

Let us suppose that there is a periodic potential, e.g., as what happens in an optical lattice,

$$U(\mathbf{x}) = U_0 \cos \mathbf{K}\cdot\mathbf{x}. \quad (4)$$

Then, the Hamiltonian density becomes

$$\mathcal{H}(\hat{\mathbf{p}},\mathbf{x},\mathbf{y}) \to \mathcal{H}_U(\hat{\mathbf{p}},\mathbf{x},\mathbf{y}) = \mathcal{H}(\hat{\mathbf{p}},\mathbf{x},\mathbf{y}) + U(\mathbf{x})\tau_3\delta^3(\mathbf{x}-\mathbf{y}). \quad (5)$$

Suppose that $\mathbf{K}$ is *not* in the *ab* plane, i.e., $\mathbf{K}\cdot\hat{\mathbf{n}}\neq 0$, and consider a cell spanned by the following vectors:

$$\mathbf{a}' = \mathbf{a} + \alpha_n\hat{\mathbf{n}},$$

$$2\mathbf{b}' = 2(\mathbf{b} + \beta_n\hat{\mathbf{n}}),$$

$$\mathbf{c}' = \frac{2\pi}{\mathbf{K}\cdot\hat{\mathbf{n}}}\hat{\mathbf{n}}, \quad (6)$$

where

$$\alpha_n = \frac{1}{\mathbf{K}\cdot\hat{\mathbf{n}}}(2\pi - \mathbf{K}\cdot\mathbf{a}),$$

$$\beta_n = \frac{1}{\mathbf{K}\cdot\hat{\mathbf{n}}}(2\pi - \mathbf{K}\cdot\mathbf{b}).$$

We see that our system has a three-dimensional periodicity represented by the commutation relations,

$$[H,T_{\mathbf{a}'}] = [H,T_{2\mathbf{b}'}] = [H,t_{\mathbf{c}'}] = 0, \quad (7)$$

where $t_{\mathbf{c}'} = \exp[i\mathbf{c}'\cdot\hat{\mathbf{p}}]$ is the usual translational operator. We also note that

$$[T_{\mathbf{a}'},T_{2\mathbf{b}'}] = [T_{2\mathbf{b}'},t_{\mathbf{c}'}] = [t_{\mathbf{c}'},T_{\mathbf{a}'}] = 0. \quad (8)$$

The Bogoliubov–de Gennes equation for our system reads

$$\int d^3y\mathcal{H}_U(\hat{\mathbf{p}},\mathbf{x},\mathbf{y})\Phi_E(\mathbf{y}) = E\Phi_E(\mathbf{x}),$$

$$\Phi_E(\mathbf{x}) = \begin{pmatrix} U_E(\mathbf{x}) \\ -V_E^*(\mathbf{x}) \end{pmatrix}, \quad (9)$$

where $[U_E(\mathbf{x}),V_E(\mathbf{x})]$ is related to the fermion operators as

$$\begin{pmatrix} \psi_\uparrow(\mathbf{x}) \\ \psi_\downarrow^*(\mathbf{x}) \end{pmatrix} = \sum_E \begin{pmatrix} U_E(\mathbf{x}) & V_E(\mathbf{x}) \\ -V_E^*(\mathbf{x}) & U_E^*(\mathbf{x}) \end{pmatrix}\begin{pmatrix} \gamma_{E\uparrow} \\ \gamma_{E\downarrow}^* \end{pmatrix},$$

and $\gamma_{E\alpha}^*$ and $\gamma_{E\alpha}$ are the creation and annihilation operators, respectively, of the Bogoliubov quasiparticles. In accordance with Eqs. (7) and (8), the solution of Eq. (9) is of the Bloch form, i.e.,

$$\Phi_{n\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}u_{n\mathbf{k}}(\mathbf{x}),$$

$$E = E_{n\mathbf{k}},$$

where n is the band index and $\mathbf{k}$ is the crystal momentum defined in the magnetic Brillouin zone,

$$-\pi/a' \le k_x < \pi/a',$$

$$-\pi/2b' \le k_y < \pi/2b',$$

$$-\pi/c' \le k_z < \pi/c',$$

and $u_{n\mathbf{k}}(\mathbf{x})$ is commensurate with the magnetic unit cell. Here, we assume the existence of a gap in the quasiparticle spectrum around zero energy.

III. THREE-DIMENSIONAL SPIN HALL CONDUCTANCE AND CHERN NUMBER

We define a spin current from the spin conservation law as

$$\dot{\rho}_s(\mathbf{x}) + \nabla\cdot\mathbf{j}_s^z = 0, \quad (10)$$

where

$$\rho_s^z(\mathbf{x}) = \frac{1}{2}\sum_{n\le 0}\int_{MBZ}\frac{d^3k}{(2\pi)^3}\Phi_{n\mathbf{k}}^\dagger(\mathbf{x})\Phi_{n\mathbf{k}}(\mathbf{x}) \quad (11)$$

is the density of the z components of the spins. The conservation is assured by the spin rotational symmetry around the z axis in the ^{3}He-A phase [Eq. (1)]. Note that in the particle-hole representation, the spin density is written by the sum of $\Phi_{n\mathbf{k}}^\dagger(\mathbf{x})\Phi_{n\mathbf{k}}(\mathbf{x})$, which has the same form as the charge density of a state $(n,\mathbf{k})$ in the usual representation. From Eqs. (10) and (11), the explicit form of the spin current for the z component is given by

$$\mathbf{J}_s^z(\mathbf{x}) = \frac{1}{2}\sum_{n\le 0}\int_{MBZ}\frac{d^3k}{(2\pi)^3}\Phi_{n\mathbf{k}}^\dagger(\mathbf{x})\frac{1}{i}[\mathbf{x},\mathcal{H}_U]\Phi_{n\mathbf{k}}(\mathbf{x}),$$

which is also of the same form as the charge current in the usual representation.

To drive the spin current, we apply an inhomogeneous magnetic field in the z direction, $\mathbf{B} = B_z\mathbf{e}_z$, with a constant gradient, i.e., $\nabla B_z = \mathrm{const}$. Then, to the Hamiltonian (5), the Zeeman coupling term $\mathcal{H}_U$ should be added, which in the particle-hole representation reads

$$\mathcal{H}_I = \frac{\mu_{\mathrm{He}}}{2}\mathbf{x}\cdot\nabla B_z. \quad (12)$$

This has the same form as the interaction between the charge density and a constant electric field in the usual representation. Then, one can see a mapping between physical quantities in the Hamiltonian for Bloch electrons in the quantum Hall (QH) system[2,5,6] and that for superfluid ^{3}He in the particle-hole representation (see Table II).

The mapping tells us that the expectation value of the spin current obtained by the Kubo formula[10] has the same form as that for the charged Hall current of 3D Bloch electrons;[2] namely,

TABLE II. The mapping between physical quantities in the Hamiltonian for Bloch electrons in the QH system (Refs. 2, 5, and 6) and that for the superfluid ^{3}He in the particle-hole representation.

Bloch electrons	Superfluid ^{3}He
e	$\frac{1}{2}$
ρ_e	ρ_s^z
$\mathbf{J}_c$	$\mathbf{J}_s^z$
$\mathbf{E}$	∇B_z

$$\langle J_{si}^z \rangle = \sigma_{ij}^s \nabla_j B_z,$$

with a conductivity

$$\sigma_{ij}^s = \mu_{\text{He}} \sum_{n\le 0,m} \int_{MBZ} \frac{d^3k}{(2\pi)^3} \times \frac{\langle u_{n\mathbf{k}}|J_{si}^z|u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|J_{sj}^z|u_{n\mathbf{k}}\rangle - (n \leftrightarrow m)}{(E_{m\mathbf{k}} - E_{n\mathbf{k}})^2}$$
$$= \frac{\mu_{\text{He}}}{4} \sum_{n\le 0} \int_{MBZ} \frac{d^3k}{(2\pi)^3 i} \epsilon_{ijk}[\nabla_{\mathbf{k}} \times \mathbf{A}_n(\mathbf{k})]_k, \qquad (13)$$

where $\nabla_{\mathbf{k}} = \partial/\partial\mathbf{k}$ and $\mathbf{A}(\mathbf{k})$ is a gauge connection in the $\mathbf{k}$ space,

$$\mathbf{A}_n(\mathbf{k}) = \langle u_{n\mathbf{k}}|\nabla_{\mathbf{k}}|u_{n\mathbf{k}}\rangle,$$

which was first introduced in this context by one of the authors.[6]

Following Ref. 2, we can relate the spin Hall conductance [Eq. (13)] to the topological Chern number[5,6] as follows. The crystal momentum is parametrized as

$$\mathbf{k} = f_{a'}\mathbf{G}_{a'} + f_{b'}\mathbf{G}_{b'}/2 + f_{c'}\mathbf{G}_{c'}, \qquad (14)$$

with $0 < f_{a'}, f_{b'}, f_{c'} \le 1$, and

$$\int_{MBZ} d^3k = \int_0^1 df_c' \frac{\mathbf{G}_{c'} \cdot \mathbf{c}'}{c'} \int_{T^2(f_{c'})} d^2k = \frac{2\pi}{c'} \int_0^1 df_{c'} \int_{T^2(f_{c'})} d^2k,$$

where $T^2(f_{c'})$ is a two-dimensional torus formed by $\mathbf{k}$'s with fixed $f_{c'}$. Similarly,

$$\int_{MBZ} d^3k = \frac{2\pi}{a'} \int_0^1 df_{a'} \int_{T^2(f_{a'})} d^2k = \frac{\pi}{b'} \int_0^1 df_{b'} \int_{T^2(f_{b'})} d^2k.$$

One may introduce a vector $\mathbf{D}$ that satisfies

$$\sigma_{ij} = \epsilon_{ijk} D_k. \qquad (15)$$

We write $\mathbf{D}$ in terms of the primitive vectors of the reciprocal lattice as

$$\mathbf{D} = \alpha\mathbf{G}_{a'} + \beta\mathbf{G}_{b'}/2 + \gamma\mathbf{G}_{c'}, \qquad (16)$$

and obtain

$$\gamma = \frac{1}{2\pi}\mathbf{c}' \cdot \mathbf{D} = -\frac{\mu_{\text{He}}}{16\pi^2} \int_0^1 df_{c'} \sum_{n\le 0} \int_{T^2(f_{c'})} \frac{d^2k}{2\pi i} [\nabla_{\mathbf{k}} \times \mathbf{A}_n(\mathbf{k})] \cdot \frac{\mathbf{c}'}{|c'|}.$$

Therefore, the integral

$$\sum_{n\le 0} \int_{T^2(f_{c'})} \frac{d^2k}{2\pi i} [\nabla_{\mathbf{k}} \times \mathbf{A}_n(\mathbf{k})] \cdot \frac{\mathbf{c}'}{|c'|} \qquad (17)$$

is the first Chern number[2,5,6] on the torus in momentum space. For the discussions of the relation between Berry's phase[11] and Eq. (17), see Refs. 1 and 12. Because of its topological nature, the integral is an integer that does not depend on $f_{c'}$ when the Fermi level for the quasiparticles is in a gap between magnetic Bloch bands. Then,

$$\gamma = -\frac{\mu_{\text{He}}}{16\pi^2} N_{c'}, \qquad (18)$$

with $N_{c'}$ an integer. We can also show that

$$\alpha = -\frac{\mu_{\text{He}}}{16\pi^2} N_{a'},$$

$$\beta = -\frac{\mu_{\text{He}}}{16\pi^2} N_{b'}, \qquad (19)$$

where $N_{a'}$ and $N_{b'}$ are integers. Finally, we have

$$\mathbf{D} = -\frac{\mu_{\text{He}}}{16\pi^2}\mathbf{G}, \qquad (20)$$

where $\mathbf{G}$ is a lattice vector reciprocal to the original lattice formed by $\mathbf{a}'$, $2\mathbf{b}'$, and $\mathbf{c}'$. Thus, the spin current is shown to be

$$\langle \mathbf{J}_s^z \rangle = \frac{\mu_{\text{He}}}{16\pi^2}\mathbf{G} \times \nabla B_z. \qquad (21)$$

This result is the spin analogy in the present system of the charged Hall current in 3DQHE.[2]

IV. SUMMARY AND DISCUSSIONS

In summary, superfluid ^{3}He-A is a good candidate system to investigate the SHE since this system has no spin-orbit couplings and has a spin rotational symmetry that ensures the spin current conservation. We have presented the 3D extension of the quantized spin Hall effect in the vortex state of a rotating ^{3}He-A superfluid film[1] in an inhomogeneous magnetic field. We have demonstrated the one to one correspondence between the SHE in ^{3}He-A in the case at hand and 3DQHE of Bloch electrons previously studied in Ref. 2 and thus have predicted the existence of the 3DQSHE in this system, when suitable conditions are satisfied.

Here, we summarize the conditions that are needed for the predicted QSHE in more detail. As we have mentioned, the ^{3}He-A superfluid has to be in the vortex phase. The vortices normally form a regular triangular lattice with the primitive vectors $\mathbf{a}=(a,0,0)$ and $\mathbf{b}=(a/2,\sqrt{3}a/2,0)$ in the plane perpendicular to the angular velocity $\mathbf{\Omega}=(0,0,\Omega)$. Then, we apply a one-dimensional periodic potential, as what happens in an optical lattice, along an axis that is not in the plane of the vortex lattice (the plane formed by vectors **a** and **b**). In the simplest case, this axis can coincide with the axis of rotation, i.e., that of $\mathbf{\Omega}$. This periodic potential should be weak and not spoil the superfluidity and the vortex phase. Then, to drive the spin current, we need an inhomogeneous static magnetic field in a fixed direction, which is coupled to the nuclear magnetic moment of the ^{3}He nuclei, with a constant gradient in its magnitude. When these conditions are satisfied, we have shown an exact analogy between the spin current of the quasiparticles in the present system (in the particle-hole representation) and the charge current of the Bloch electrons in a three-dimensional periodic potential. Based on the knowledge of the latter[2] with a connection to the Diophantine equations (see Ref. 5 and the third paper in Ref. 2), a three-dimensional quantized spin Hall effect with nonzero Hall conductances is generically expected, when the Fermi level of the Bogoliubov quasiparticles just completely fill a certain number of Bloch bands, opening a gap in the spectrum of quasiparticles.

Since the vortex lattice is created by rotation, the vortex structure near the boundary could be rather complicated. How spin accumulation would be affected by conditions at the boundary is an important issue for experimental detection. Theoretically, this is a separate issue from the bulk effect we have addressed in this paper and is certainly worth further study.

Finally, we emphasize that in the above, we have mentioned the ^{3}He-A phase as a familiar example of a p-wave paired Fermi system. Actually, the above discussions are also valid for cold Fermi atomic gases that allow p-wave pairing in a certain parameter regime. This remark opens the door for more candidate systems that may experimentally realize the predicted QSHE.

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